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Optimal analytic extrapolations revisited

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Abstract. The problem of optimal analytic extrapolation of holomorphic functions from a finite set of interior data points to another interior point is completely solved in the general case of data known with unequal errors. Simple and easy to handle algorithms are obtained.

1. Introduction

In the last fifteen years there has been an explosion of interest in the study of analytic extrapolations, many functions of physical interest being known to have analytic properties. Consequently much effort has been devoted to describing such extrapolations (Goebel 1958, Chew and Low 1959, Frazer 1961, Ciulli and Fischer 1961, Cutkosky and Deo 1968a, b, Pisut and Presnajder 1969a, b, 1970, Pietarinen 1972, 1973, Cutkosky 1973, Ciulli and Nenciu 1973, Ciulli *et al* 1975, Stefanescu 1980, Ciulli and Spearman 1982). Typically such a function is determined within the holomorphy domain at a finite number of points. Elsewhere inside the analyticity domain, but outside the physical region of the process considered, the values of the function may be of considerable interest. Examples are provided by the production processes $PN \to M^*B$ (P for pseudoscalar mesons π and K, and B for baryons N and Δ) which afford information on $\pi\pi$ and πK scattering. This information, unlike that for hadronic processes, cannot be obtained directly from experiment, a reliable picture of these processes emerging only by using various indirect methods. These methods make substantial use of analyticity properties of production amplitudes, the information on $\pi\pi$ and πK being obtained by some kind of analytic extrapolation.

For definiteness we shall consider such a production process. Its amplitude $F(t)$ is a meromorphic function in the complex t-plane with a cut $9m^2 \leqslant t \leqslant \infty$, where t is the square of the difference of the initial and final baryon four momenta and m is the pion mass. Certainly, the amplitude also depends upon other kinematical variables, but we singled out t since it is of interest in what follows. The residue at the pion pole $t = m^2$ gives the scattering amplitude for the process $\pi P \to \pi P$. The experimental data are available at a finite number of points inside the analyticity region. Since the point $t = m^2$ is not accessible to experiment we have to extrapolate the quantity $(t - m^2)F(t)$ which is an analytic function.

Until now the extrapolation was done essentially by two methods. The first one was proposed by Goebel (1958) and Chew and Low (1959). It consists in fitting the small-t production data for $F(t)$ to a polynomial in t, and using that to extrapolate to $t = m^2$. It was observed by Frazer (1961) that the data are better fitted by a polynomial in $z(t)$, instead of a polynomial in t, where $z(t)$ is a function that maps the analyticity domain onto the unit disc. This proposal of Frazer has been improved by Cutkosky and Deo (1968a, b) and Ciulli (1969a, b) who found the conformal mappings leading to optimal asymptotic convergence of polynomial expansions.

The second method of extrapolation uses the interpolation theory of analytic functions to bring the problem to a standard Pick–Nevanlinna problem.

By the conformal mapping

$$z(t)=[1-(1-t/9m^2)^{1/2}]/[1+(1-t/9m^2)^{1/2}] \tag{1.1}$$

the cut t-plane is mapped onto the unit disc D, $D=\{z, |z|<1\}$, the real axis $-\infty \leqslant t \leqslant 9m^2$ is mapped onto the real segment $-1 \leqslant z \leqslant 1$ and the pole at $t=m^2$ is mapped to z_0, $0<z_0<1$.

Many authors have noticed that the analytic extrapolation is an ill posed problem, and in order to obtain reliable results a stability condition has to be imposed since without it any value of the extrapolate can be obtained (Ciulli 1969a, b, Pisut 1970, Ciulli *et al* 1975, Atkinson 1978).

For our purposes it is sufficient to have an upper bound upon the modulus of the function along the cuts, of the form

$$\left|\frac{z-z_0}{1-z_0 z}F(t(z))\right| \leqslant M(\theta), \qquad z=\exp(\mathrm{i}\theta). \tag{1.2}$$

Let us define

$$g(z)=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\ln M(\theta)\,\mathrm{d}\theta\right)$$

which is an analytic function without zeros inside the unit disc D and whose modulus satisfies

$$|g(\mathrm{e}^{\mathrm{i}\theta})|=M(\theta). \tag{1.3}$$

We shall denote by $f(z)$ the reduced amplitude

$$f(z)=\frac{z-z_0}{1-z_0 z}\,\frac{F(t(z))}{g(z)} \tag{1.4}$$

and, owing to (1.2) and (1.3), $f(z)$ is a bounded analytic function, i.e. $|f(z)|<1$ for $z \in D$. We shall denote by w_i and δ_i respectively the values of the function $f(z)$ at the points z_i, $i=1,\ldots,n$, and their errors.

In general the problem of extrapolation consists in finding $M=\sup f(z_0)$ and $m=\inf f(z_0)$ in the class of real analytic functions $\overline{f(z)}=f(\bar{z})$, satisfying $|f(z_i)-w_i| \leqslant \delta_i$, $i=1,\ldots,n$. The optimal extrapolated value and its error are given by $f(z_0)=(M+m)/2$ and $E(z_0)=(M-m)/2$ respectively.

The Pick–Nevanlinna theory provides a straightforward solution of the problem when the data are known exactly, without errors.

The problem is solvable if and only if the values w_i fulfil a consistency condition (Krein and Nudelman 1973), i.e. the positive definiteness of the matrix

$$\Lambda=\left(\frac{1-w_i w_j}{1-z_i z_j}\right)_{i,j=1}^{n}. \tag{1.5}$$

If Λ has negative eigenvalues the problem has no solution. The set of all solutions can be found by applying Schur's algorithm (Walsh 1960, Atkinson 1978) or Krein's elegant method (Krein and Nudelman 1973). For practical purposes we suggest the use of the first method which is constructive.

The solutions of the problem: find an analytic function $f(z)$, $|f(z)| \leq 1$ for $z \in D$ such that $f(z_i) = w_i$, $i = 1, \ldots, n$, are given by $f(z) = p_0(z)/q_0(z)$ where $p_0(z)$ and $q_0(z)$ are the solutions of the recurrence relations

$$\begin{aligned} p_{m-1}(z) &= [(z - z_m)/(1 - z_m z)]p_m(z) + w_m^{(m-1)} q_m(z), \\ q_{m-1}(z) &= q_m(z) + w_m^{(m-1)} p_m(z)(z - z_m)/(1 - z_m z), \\ m &= 1, 2, \ldots, n, \end{aligned} \tag{1.6a}$$

$p_n(z)$ is an arbitrary bounded analytic function, $|p_n(z)| \leq 1$, $q_n(z) = 1$ and

$$w_k^{(m)} = \frac{1 - z_k z_m}{z_k - z_m} \frac{w_k^{(m-1)} - w_m^{(m-1)}}{1 - w_k^{(m-1)} w_m^{(m-1)}}, \tag{1.6b}$$

$k = m + 1, \ldots, n$; $m = 1, 2, \ldots, n$; $w_i^{(0)} = w_i$, $i = 1, \ldots, n$.

The upper and lower bounds M and m are obtained for $p_n(z) = \pm 1$ respectively.

The error $E(z_0) = (M - m)/2$ has nothing to do with the experimental errors that we supposed to be equal to zero, being solely a consequence of the freedom allowed by the stabilising condition (1.2).

It the errors are not zero both extrapolation methods do not provide a reliable estimate of the effect of these errors upon the extrapolated values, or an algorithm to calculate directly the extrapolated value in terms of w_i. The recently proposed χ^2-test by Stefanescu (1980) does not improve the situation.

The aim of this paper is to give a simple solution to both these problems by presenting a new algorithm for analytic extrapolations. It is shown that the optimal extrapolation of bounded analytic functions can be done by linear methods, i.e. methods of the form

$$f(z_0) = \sum_{i=1}^{n} C_i(z_0) w_i$$

where the weights $C_i(z_0)$ depend upon the errors and not upon the function to be extrapolated.

The method provides also a simple algorithm for the calculation of the error of the extrapolated value.

Our results extend to the case of unequal errors those obtained previously by Osipenko (1982).

The paper is organised as follows. In § 2 we present the main theorems which will be used throughout the paper and rederive Osipenko's results (Osipenko 1976) for data known exactly, without errors. Section 3 contains the optimal extrapolation and error calculation formulae. In § 4 the method is applied to a model set of data with controllable errors. Some concluding remarks are given in § 5. The proofs of the theorems are collected in the appendix.

2. Main theorems

Let us define the classes of real analytic functions

$$\begin{aligned} B &= \{f(z) \,|\, |f(z)| \leq 1, f(\bar{z}) = \overline{f(z)} \text{ for } z \in D\}, \\ A &= \{f(z) \,|\, f \in B, |f(z_i) - w_i| \leq \delta_i, i = 1, \ldots, n\}, \\ AB &= \{f(z) \,|\, f \in B, |f(z_i)| \leq \delta_i, i = 1, \ldots, n\}, \end{aligned}$$

where z_i and w_i are real numbers, $z_i, w_i \in (-1, 1)$ and $1 \geq \delta_i \geq 0$, $i = 1, \ldots, n$.

We shall suppose that A is a non-void set. A sufficient condition for this is that the Λ-matrix (1.5) should be strictly positive.

We shall use the following notations: $\delta = (\delta_1, \ldots, \delta_n)$, $w = (w_1, \ldots, w_n)$ and $E = (z_1, \ldots, z_n)$.

By an extrapolation formula of $f(z)$ to the real point $z = z_0$, we mean any arbitrary linear or nonlinear formula $S(z_0, w, \delta)$ and we say that quantity

$$\sup_{f \in A} |f(z_0) - S(z_0, w, \delta)|$$

is the error of the extrapolation.

The best extrapolation formula $S^0(z_0, w, \delta)$ will be that for which this error is least, i.e. for which the following infimum is attained;

$$\begin{aligned} \varepsilon(z_0, E, \delta) &= \sup_{f \in A} |f(z_0) - S^0(z_0, w, \delta)| \\ &= \inf_S \sup_{f \in A} |f(z_0) - S(z_0, w, \delta)|. \end{aligned} \tag{2.1}$$

Any extrapolation problem, analytic or not, has two steps. The first consists in finding the optimal formula for $f(z_0)$ in terms of the given data z_i, w_i, δ_i, $i = 1, \ldots, n$, by a linear method if possible; the second is the calculation of the error (2.1).

With the previous notations we have the following theorem.

Theorem 1. For any real z_0, $-1 < z_0 < 1$:

(a) $$\varepsilon(z_0, E, \delta) = \inf_S \sup_{f \in A} |f(z_0) - S(z_0, w, \delta)| = \sup_{f \in AB} |f(z_0)|. \tag{2.2}$$

(b) There is a linear method of extrapolation

$$f(z_0) = \sum_{j=1}^{n} C_j(z_0, \delta, E) w_j \tag{2.3}$$

which saturates the above equality, i.e.

$$\sup_{f \in A} \left| f(z_0) - \sum_{j=1}^{n} C_j(z_0, \delta, E) w_j \right| = \sup_{f \in AB} |f(z_0)|.$$

(c) If the function $\varphi_j(\alpha, \delta) = \sup_{f \in A_j} f(z_0)$ where

$$A_j = \{f(z) \,|\, f \in B, |f(z_j) - \alpha| \leqslant \delta_j, |f(z_i)| \leqslant \delta_i, i \neq j, i = 1, \ldots, n\}$$

is differentiable for $\alpha = 0$, then $C_j(z_0, \delta, E)$ are uniquely defined by

$$C_j(z_0, \delta, E) = \mathrm{d}\varphi_j(\alpha, \delta)/\mathrm{d}\alpha|_{\alpha=0}. \tag{2.4}$$

This proposition provides us with the tools of optimal analytic extrapolations from interior data points to another interior point for bounded analytic functions within the class defined by (1.2). The complicated problem of the evaluation of the error (2.1) reduces to a simpler one, that of finding the extremals of $\sup_{f \in AB} |f(z_0)|$.

The most important peculiarity of our method worth noticing is that the extrapolated value (2.3) is calculated linearly from the input values w_i. A proof of this result is given in the appendix.

A second result which will be used in the following is a theorem by Hejhal (1974) which covers a very large class of extremal problems. We shall state it in the form we need but it is more general.

Theorem 2. Let us suppose that in the problem $\sup_{f\in A}|f(z_0)|$ there is at least one extremal $f^*(z)\neq$ constant. Then

$$|f^*(z_i)-w_i|=\delta_i,\ i=1,\ldots,n. \tag{2.5}$$

for almost all indices i.

In many problems of best approximation an information 'super-abundance' phenomenon may be present. This effect is well known in the theory of Chebyshev approximation. A phenomenon of this type is also present in optimal analytic extrapolation. It is a reflection of the fact that the analyticity and the boundedness imply strong correlations between the values taken by the analytic functions at distinct points. The extremals of the problems will verify (2.5) for some values of the subscript i; for other values they will satisfy the strict inequalities $|f^*(z_j)-w_j|<\delta_r$.

Theorem 2 gives the description of the behaviour of extremals and is of great use in finding them. Its content may also be interpreted as follows: not all data points carry useful information and we have to discard these points when doing extrapolations.

We shall now apply these results to finding the optimal formula of extrapolation and the error, for data known exactly, i.e. $\delta_i=0$, $i=1,\ldots,n$. We define the Blaschke products

$$B(z)=\prod_{i=1}^{n}\frac{z-z_i}{1-z_iz},\qquad B_j(z)=\prod_{\substack{i=1\\ i\neq j}}^{n}\frac{z-z_i}{1-z_iz},\qquad j=1,\ldots,n. \tag{2.6}$$

According to theorem 1 the error is given by

$$\varepsilon(z_0,E,\delta)=\sup_{f\in AB}|f(z_0)|. \tag{2.7}$$

Since $\delta_i=0$, AB will be the set

$$AB=\{f(z)\,|\,f\in B, f(z_1)=\ldots=f(z_n)=0\}.$$

It is easily seen that the functions of the class whose supremum is sought in (2.7) have the form $f(z)=B(z)h(z)$, where $h(z)\in B$ is an arbitrary bounded analytic function. Therefore the extremals of the problem (2.7) have the form $f^*(z)=\pm B(z)$ and consequently

$$\varepsilon(z_0,E,\delta)=|B(z_0)|. \tag{2.8}$$

The weights $C_j(z_0,\delta,E)$ are the derivatives of the extremals of

$$\varphi_j(\alpha)=\sup_{f\in A_j}f(z_0) \tag{2.9}$$

where A_j is the following set:

$$A_j=\{f(z)\,|\,f\in B, f(z_i)=\alpha\delta_{ij}, i=1,\ldots,n\}.$$

The function $f(z)=B_j(z)g(z)$, $g(z)\in B$ will be in the class whose supremum is sought in (2.9) if $g(z_j)=\alpha/B_j(z_j)$. The Schur algorithm (1.6) gives the generic form of a function $f\in A_j$

$$f(z)=\frac{[(z-z_j)/(1-z_jz)]h(z)+\alpha/B_j(z_j)}{1+\alpha h(z)[(z-z_j)/(1-z_jz)]/B_j(z_j)}B_j(z)$$

where $h(z)\in B$ is an arbitrary bounded analytic function. The extremal of the problem

(2.9) is obtained for $h(z)=1$, hence

$$\varphi_j(\alpha)=B_j(z_0)\frac{(z_0-z_j)/(1-z_jz_0)+\alpha/B_j(z_j)}{1+\alpha[(z_0-z_j)/(1-z_jz_0)]/B_j(z_j)}.$$

Therefore

$$C_j(z_0,\delta,E)=\frac{\mathrm{d}\varphi_j(\alpha)}{\mathrm{d}\alpha}\bigg|_{\alpha=0}=\frac{B_j(z_0)}{B_j(z_j)}\left[1-\left(\frac{z_0-z_j}{1-z_jz_0}\right)^2\right].$$

The optimal extrapolation formula has the form

$$f(z_0)=(1-z_0^2)\sum_{j=1}^{n}\frac{B_j(z_0)}{B_j(z_j)}\frac{1-z_j^2}{(1-z_0z_j)^2}w_j. \tag{2.10}$$

The above results have been obtained by Osipenko (1976); they have a very simple form in contrast with those given by the nonlinear Pick–Nevalinna method.

3. Optimal formula of extrapolation and calculation of error

Theorem 1 reduces the problem of best extrapolation to that of solving two extremal problems. Since the functions we deal with are real analytic functions and we extrapolate from interior real points to another real point z_0, we may normalise the extremals $f^*(z)$ of the above problems such that $f^*(z_0)>0$.

We shall proceed by finding the algorithm for the calculation of the error $\varepsilon(z_0,E,\delta)$. It is given by

$$\varepsilon(z_0,E,\delta)=\sup_{f\in AB}|f(z_0)| \tag{3.1}$$

and in order to calculate it explicitly we have to construct the extremal of the problem (3.1). This extremal, according to theorem 2, will satisfy $f^*(z_i)=\pm\delta_i$ for almost all i. Actually the following proposition holds.

Theorem 3. The function

$$f^*(z)=\lambda\prod_{j=1}^{m}\frac{z-\alpha_j}{1-\alpha_jz}, \qquad \alpha_j\in(-1,1),$$

normalised by the condition $f^*(z_0)>0$, is an extremal of the problem (3.1) if and only if for any $z_i\in E=\{z_1,\ldots,z_n\}$, $|f^*(z_i)|\leqslant\delta_i$ and there does exist a subset $F=\{z_{i_1},\ldots,z_{i_m}\}$, $F\subset E$ such that

$$f^*(z_{i_k})=\begin{cases}(-1)^{p+k}\delta_{ik}, & k=1,\ldots,p,\\ (-1)^{p+k+1}\delta_{i_k}, & k=p+1,\ldots,m,\end{cases} \tag{3.2}$$

where p is an integer such that $z_{i_p}<z_0<z_{i_{p+1}}$, $\lambda=(-1)^{m+p}$. Here we have the convention $z_{i_0}=-1$, $z_{i_{m+1}}=1$.

This theorem is a reformulation of a theorem found by Osipenko (1982) for the case of data known with equal errors $\delta_i=\delta$, $i=1,\ldots,n$. The proof of this result may be given along Osipenko's lines, the only change being the use of theorem 2 instead of Heins' theorem (Heins 1945, Havinson 1963) used by Osipenko, so that we shall omit it.

The first problem to be solved is the finding of the set F. If F is strictly contained in E we have to discard some points. This means that there will be points z_j where the extremal of the problem (3.1) will satisfy the strict inequalities $|f^*(z_j)| < \delta_j$. Since there is no *a priori* reason to discard any point, we form the Λ-matrix (1.5) in order to see if the data (3.2) are compatible with the analyticity and boundedness properties of the extremal.

Let us suppose, for definiteness, that we have the ordering $-1 < z_1 < \ldots < z_n < z_0 < 1$. Then Λ has the form

$$\Lambda = \left(\frac{1-(-1)^{i+j}\delta_i\delta_j}{1-z_iz_j}\right)_{i,j=1}^{n}. \tag{3.3}$$

The factor $(-1)^{i+j}$ originates in the relations (3.2) which show that the sign alternates from one point to the next. If $\Lambda > 0$ (all eigenvalues positive) we do not discard any point and F is identical to the initial set E. If Λ has negative or zero eigenvalues we have to discard points one after the other until Λ becomes positive. Since we are interested in obtaining the smallest possible error for the extrapolated value, we suggest to discard first the data points which have the greatest errors. Once F is determined, we make the notations

$$z_{i_k} = u_k, \qquad w_{i_k} = \omega_k, \qquad \delta_{i_k} = \varepsilon_k, \qquad k = 1, \ldots, m. \tag{3.4}$$

Let $\tilde{B}_j(z)$ and $\tilde{B}(z)$ denote the Blaschke products (1.6) constructed with the set $F = \{u_1, \ldots, u_m\}$. The optimal formula of extrapolation and the error are given by the following theorem.

Theorem 4. Let $F = \{u_1, \ldots, u_m\} \subset E$ be the set determined in theorem 3. Then the method

$$f(z_0) = (1 - z_0^2) \sum_{j=1}^{m} \frac{\tilde{B}_j(z_0)}{\tilde{B}_j(u_j)} \frac{1-u_j^2}{(1-u_jz_0)^2} a_j^2 \omega_j \tag{3.5}$$

is the optimal method of extrapolation. The error of this method is given by

$$\varepsilon(z_0, E, \delta) = p_{10}(z_0)/q_{10}(z_0) = \ldots = p_{m0}(z_0)/q_{m0}(z_0) \tag{3.6}$$

where $a_j = q_{j0}(u_j)/q_{j0}(z_0)$ and $p_{j0}(z)$ and $q_{j0}(z)$ are the solutions of the recurrence relations

$$\begin{aligned} &p_{jk-1}(z) = b_{jk}(z)p_{jk}(z) + d_{jk}^{(k-1)}q_{jk}(z), \\ &q_{jk-1}(z) = q_{jk}(z) + d_{jk}^{(k-1)}b_{jk}(z)p_{jk}(z), \\ &k = 1, \ldots, m, \\ &p_{jm}(z) = \operatorname{sgn} \tilde{B}(z_0), \qquad q_{jm}(z) = 1. \end{aligned} \tag{3.7}$$

Here

$$\begin{aligned} &d_{jk}^{(l)} = b_{jl}(u_{jk})(d_{jk}^{(l-1)} - d_{jl}^{(l-1)})/(1 - d_{jk}^{(l-1)}d_{jl}^{(l-1)}), \\ &k = l+1, \ldots, m, \qquad l = 1, \ldots, m, \\ &d_{jk}^{(0)}, u_{jk}, b_{jk}(z) = \begin{cases} d_k, u_k, (1-u_kz)/(z-u_k), & k \neq j, m, \\ d_m, u_m, (1-u_mz)/(z-u_m), & k = j, \\ d_j, u_j, (1-u_jz)/(z-u_j), & k = m, \end{cases} \\ &d_l = \varepsilon_l \operatorname{sgn}[\tilde{B}_l(z_0)/\tilde{B}_l(u_l)], \qquad l = 1, \ldots, m. \end{aligned}$$

A proof of this theorem is given in the appendix.

The formulae (3.5) and (3.6) completely solve the problem of optimal extrapolation for data known with unequal errors.

However, the problem of extrapolation of analytic functions for which a bound like (1.2) is not available remains unsolved. There are only a few cases when we know an upper bound along the cuts on the function modulus. These favourable cases include the electromagnetic form factors and the hadronic contribution to the muon anomalous magnetic moment. In some cases the theoretical models may provide bounds, like Froissart's bound. These asymptotic bounds are generally known up to multiplicative constant or smoothly variable factors. Even so they are of interest in our problem since we can construct the reduced amplitudes (1.4) ignoring the unknown factors.

Thus we may suppose that $|f(z)| \leq M$ for $z \in D$, even if we do not know what value M takes. In such cases the measured values and the corresponding errors set a scale for the bound and we have to find out some means of extracting this information from the data.

Theorem 1 is still true such that the error and the weights C_j are determined by solving the corresponding extremal problems. The key proposition is again theorem 2 which describes the behaviour of the extremals. In such cases we do not discard any point. Let us suppose the ordering $-1 < z_1 < \ldots < z_n < z_0 < 1$. The problem is to find the minimal norm of an analytic function that satisfies $f(z_i) = (-1)^{n-i}\varepsilon_i$, $i = 1, \ldots, n$, where ε_i are the values of the errors. The norm we look for is the H^∞-norm, i.e. $\|f\| = \sup_{r<1} |f(r\,e^{i\theta})|$.

It is well known that for a finite set of data points there is always a bounded analytic function that takes prescribed values v_j at the points z_j, $f(z_j) = v_j$, $j = 1, \ldots, n$ (Walsh 1960).

The minimal norm is given by the greatest positive root of the equation

$$\det\left(\frac{\lambda^2 - (-1)^{i+j}\varepsilon_i\varepsilon_j}{1 - z_i z_j}\right)^n_{i,j=1} = 0.$$

This minimal norm M_0 can be used for doing extrapolations. The error and the extrapolated value are given by formulae like (3.5) and (3.6), the only change being $m \to n$ and $d_{jk}^{(0)} \to d_{jk}^{(0)}/M_0$, $j, k = 1, \ldots, n$.

4. A numerical example

We shall now apply our method to a model set of data with controllable errors. In the simple example we consider, the data are generated by the function

$$f(z) = [(z + 0.2)/(1 + 0.2z)] \exp[-\tfrac{1}{2}(1 + z^2)].$$

The 'experimental' points belongs to the segment $(-0.95, 0.25)$. We shall now list some typical situations. The data points are $E = (-0.95, -0.87, -0.7, -0.63, -0.5, -0.35, -0.15, 0.0, 0.1, 0.25)$ and $z_0 = 0.8$.

The exact value of the function is $f(0.8) = 0.3797$, by comparison with the extrapolated value $f(0.8) = 0.3558 \pm 0.265$ obtained by taking all the errors equal to zero. If we ascribe a 5% error to every exact value of the function at the above points, the data for the error calculation are not compatible. They become again compatible after discarding six of them. The result is $f(0.8) = 0.28 \pm 0.555$. We want to notice the effect of the errors on the final result. If z_0 is closer to the experimental region the situation improves. For example with $z_0 = 0.35$ one gets $f(0.35) = 0.2925 \pm 0.052$ and the exact value is $f(0.35)=0.2932$. If z_0 is lying among the experimental points the results are impressive, the effect of the errors being drastically suppressed. We obtain, for example, $f(-0.4)=-0.12\pm0.008$ by comparison with the exact value $f(-0.4)=-0.1217$.

In conclusion the method works very well and reliable results are obtained only by taking into account the propagation of the errors.

5. Conclusion

In this paper we have been concerned with the problem of optimal analytic continuation from a finite set of data points inside the analyticity domain to another interior point when the values of the analytic function are known with errors.

This problem emerged in connection with the finding of an alternative to the existing methods of extrapolation which are known to be unsatisfactory, especially as concerns the evaluation of the error of the extrapolated values. The new algorithm is simple enough and easy to handle. It can be applied even in those cases when no explicit upper bound on the modulus of the amplitude is known.

The obtained formulae provide a method of estimation of the function values for the entire real segment $(-1, 1)$, i.e. both inside and outside the physical domain. Thus one can obtain curves affected by known errors for physical quantities for all real values of the transverse momentum t inside the holomorphy domain. This will help in comparing data with theoretical models even for values of t outside the physical region.

Appendix

In the following $f(z)$ will denote a real function defined on $(-1, 1)$ which takes values in the same interval. We can limit ourselves to this class since the functions we are dealing with are real analytic functions and we extrapolate from a real set of data points to another real point. In this way the proof of theorem 1 is simplified.

Proof of theorem 1.
Let Y be the closure of the $(n+1)$-dimensional set $(y_0, y_1, \ldots, y_n)=(f(z_0), w_1, \ldots, w_n)$ where $f(z)\in A$, $-1<z_0<1$ and $w_i\in(-1, 1)$, $i=1, \ldots, n$ are real numbers. The classes A, B and AB have been defined in § 2. Y is a convex body.

Let us denote

$$C_0(\delta)=\sup_{(y_0,0,\ldots,0)} y_0. \tag{A1}$$

We shall prove that $C_0(\delta)=\sup_{f\in AB}|f(z_0)|$. The relation (A1) implies

$$C_0(\delta)=\sup_{\substack{f\in A\\ w_1=\ldots=w_n=0}} f(z_0)=\sup_{\substack{f\in B\\ |f(z_i)|\leq\delta_i, i=1,\ldots,n}}=\sup_{f\in AB} f(z_0).$$

The last part of the above relation is a consequence of the constraints $|f(z_i)-w_i|\leq\delta_i$, $i=1, \ldots, n$.

$f(z_0)$ is real since z_0 is real and $\overline{f(z)}=f(\bar{z})$. Also $f\in AB\Rightarrow -f\in AB$. Hence $\sup_{f\in AB} f(z_0)=\sup_{f\in AB}|f(z_0)|$.

Through the point $(C_0(\delta), 0, \ldots, 0)$ which is on the frontier of the convex set Y one can construct a supporting hyperplane whose equation has the form

$$y_0 = \sum_{j=1}^{n} C_j(\delta) y_j + C_0(\delta).$$

Since Y is symmetric with respect to the origin of coordinates

$$y_0 = \sum_{j=1}^{n} C_j(\delta) y_j - C_0(\delta)$$

will also be a supporting hyperplane for Y. In fact Y is contained between these two hyperplanes. Thus

$$\left| y_0 - \sum_{j=1}^{n} C_j(\delta) y_j \right| \leqslant C_0(\delta)$$

or in another form

$$\left| y_0 - \sum_{j=1}^{n} C_j(\delta) w_j \right| \leqslant C_0(\delta)$$

a relation valid for any $f \in A$. We obtain

$$\sup_{f \in A} \left| f(z_0) - \sum_{j=1}^{n} C_j(\delta) w_j \right| \leqslant C_0(\delta). \tag{A2}$$

The relation (A1) implies that for any $\varepsilon > 0$ there is a $f_\varepsilon \in A$ such that

$$|f_\varepsilon(z_0)| > C_0(\delta) - \varepsilon \tag{A3}$$

and $w_1 = \ldots = w_n = 0$. But the set A with the conditions $w_1 = \ldots = w_n = 0$ coincides with the set AB. Thus $f_\varepsilon \in AB$. $-f_\varepsilon$ will have the same property (A3).

Now let S be any arbitrary method of approximation of the value $f(z_0)$, $f \in A$. Then for $\pm f_\varepsilon \in AB$ the approximate value of $f(z_0)$ will be equal to

$$S(\delta, w_1, \ldots, w_n) = S(\delta, 0, \ldots, 0) = S^0.$$

But the inequaliites

$$|f_\varepsilon(z_0) - S^0| + |-f_\varepsilon(z_0) - S^0| \geqslant |f_\varepsilon(z_0) - S^0 + S^0 - (-f_\varepsilon(z_0))| > 2(c_0(\delta) - \varepsilon)$$

imply that for every function $\pm f_\varepsilon$ the approximate error is greater than $C_0(\delta) - \varepsilon$ and since ε is arbitrary we obtain

$$\inf_{S} \sup_{f \in A} |f(z_0) - S(z_0, w, \delta)| \geqslant C_0(\delta). \tag{A4}$$

The inequalities (A2) and (A4) show that there is a linear method of extrapolation. It is easily seen that

$$C_j(\delta) = \partial y_0 / \partial y_j |_{y_1, \ldots, y_n = 0}, \qquad j = 1, \ldots, n.$$

Thus we have the following proposition. Let us denote $\varphi_j(\varepsilon, \delta) = \sup_{f \in A_j} f(z_0)$ where $A_j = \{f \mid f \in B, |f(z_i) - \varepsilon \delta_{ij}| \leqslant \delta_i, i = 1, \ldots, n\}$. If $\varphi_j(\varepsilon, \delta)$ is differentiable in $\varepsilon = 0$, $C_j(\delta) = \mathrm{d}\varphi_j(\varepsilon, \delta)/\mathrm{d}\varepsilon|_{\varepsilon=0}$.

Proof of theorem 4. Theorem 3 allows us to find the set $F = (u_1, \ldots, u_m) \subset E$, i.e. the maximal number of data points whose errors are compatible with the boundedness and analyticity properties of the extremals. The point z_0 to which we extrapolate is supposed to be outside the physical region $-1 < u_1 < \ldots < u_m < z_0 < 1$, the other cases being treated similarly. F is determined by the condition that all eigenvalues of the matrix

$$\Lambda = \left(\frac{1-(-1)^{i+j}\varepsilon_i\varepsilon_j}{1-u_iu_j} \right)_{i,j=1}^{m} \tag{A5}$$

are positive. The notations are those of § 3.

The strict positivity of Λ implies the existence of a continuum of solutions (Krein and Nudelman 1973) to the following problem of interpolation: find a bounded analytic function $f_j^*(z) \in B$ such that

$$\begin{aligned} &f_j^*(u_k) = (-1)^{m-k}\varepsilon_k, \\ &f_j^*(u_j) = (-1)^{m-j}\varepsilon_j + \alpha; \qquad k \neq j;\ k = 1, \ldots, m, \end{aligned} \tag{A6}$$

where $|\alpha|$ is sufficiently small.

The interpolation problem (A6) can be solved by applying Schur's algorithm (1.6). The order in which the points u_i are introduced into the algorithm is arbitrary and we have made use of it by permuting the points u_j and u_m in order to simplify the subsequent calculations.

d_{jk}^{l}, u_{jk} and $b_{jk}(z)$ are defined as in theorem 4, the only change being that

$$d_{jk}^{(0)} = \begin{cases} (-1)^{m-k}\varepsilon_k, & k \neq j, m, \\ \varepsilon_m, & k = j, \\ (-1)^{m-j}\varepsilon_j + \alpha, & k = m. \end{cases}$$

Thus $\varphi_j(z, \alpha) = p_{j0}(z, \alpha)/q_{j0}(z, \alpha)$ where p_{j0} and q_{j0} are the solutions of (1.6*a*), and w_i are given by (A6). The dependence upon α of $\varphi_j(z, \alpha)$ is contained only in the coefficients $d_{jm}^{(m-1)}$. Thus we obtain

$$\frac{\mathrm{d}\varphi_j(z_0, \alpha)}{\mathrm{d}\alpha} = \frac{(1-z_0^2)(1-u_j^2)}{(1-z_0u_j)^2}\tilde{B}_j(z_0)\frac{D(\alpha)}{q_{j0}^2(z, \alpha)}$$

where

$$D(\alpha) = \frac{\partial d_{jm}^{(m-1)}}{\partial \alpha}\prod_{k=1}^{m-1}[1-(d_{jk}^{(k-1)})^2].$$

Since $\varphi_j(u_j, \alpha) = (-1)^{m-j}\varepsilon_j + \alpha$ we obtain $\mathrm{d}\varphi_j(u_j, \alpha)/\mathrm{d}\alpha = \tilde{B}_j(u_j)D(\alpha)/q_{j0}^2(u_j, \alpha) = 1$. Hence $D(\alpha) = q_{j0}^2(u_j, \alpha)/\tilde{B}_j(u_j)$ and consequently

$$\frac{\mathrm{d}\varphi_j(z_0, \alpha)}{\mathrm{d}\alpha} = \frac{\tilde{B}_j(z_0)}{\tilde{B}_j(u_j)}\frac{(1-z_0^2)(1-u_j^2)}{(1-z_0u_j)^2}\frac{q_{j0}^2(u_j, \alpha=0)}{q_{j0}^2(z_0, \alpha=0)}.$$

In the last formula

$$d_{jm}^{(0)}(\alpha=0) = (-1)^{m-j}\varepsilon_j.$$

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